

United States Naval Postgraduate School



VIBRATION ANALYSIS OF CYLINDRICAL
SHELLS BY SEVERAL FINITE DIFFERENCE SCHEMES

by

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June 1971

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ABSTRACT:

Several finite differencing schemes are used to compute the natural frequencies and mode shapes of simply supported circular cylindrical shells in an attempt to determine the most accurate numerical model. Sets of difference equations are developed from the governing differential field equations and by minimizing the finite difference form of the Lagrangian energy function. Both finite differences and trigonometric expansions are used to model the circumferential behavior. Staggered or half-stations are used in addition to the conventional differencing schemes. The results indicate that the schemes using the trigonometric expansions are generally more accurate than those using finite differences for the circumferential derivatives. Furthermore, the conventional differencing scheme is shown to be as accurate as the half-station scheme when the field equation approach is used in conjunction with the trigonometric expansions.

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INTRODUCTION

There are several large-scale digital computer programs presently in operation that use the method of finite differences to predict the natural frequencies and mode shapes of general shell of revolution structures [1,2,3]. These programs use a variety of numerical schemes in the development of the governing finite difference equations. This paper uses several of these schemes, and some others, to compute the natural frequencies and modes of simply supported circular cylindrical shells. Donnell's theory is employed, and the unknown dependent variables are the three displacements. Two fundamental approaches are considered in the development of the difference equations; one is an energy formulation in which the variation of the finite difference form of the Lagrangian energy function with respect to the displacement variables at each station is set equal to zero, the other uses finite difference approximations to the continuous derivatives in the governing field equations. Within these two approaches, both finite differences and trigonometric expansions are used to model the circumferential behavior. In addition, the three displacements are either defined at the same stations, as in the conventional finite difference method, or they are defined at different, or staggered, stations. The

^{*}This work was sponsored in part by the Foundation Research Program at the Naval Postgraduate School.

latter technique is also known as the half-station method, and has been used in the static analysis of beam-columns [4] and cylindrical shell roofs [5], and in the vibration analysis of circular rings [6]. The results presented in references [4] - [6] indicate that the staggered station scheme yields solutions that are significantly more accurate than the solutions from the conventional, or whole-station, method. The goal of this paper is to determine the most accurate numerical model from among the several considered for predicting the natural frequencies and mode shapes of simply supported circular cylindrical shells. The applicability of that scheme to the general shell of revolution is also examined.

DIFFERENTIAL EQUATIONS, NATURAL MODES, AND NATURAL FREQUENCIES

Consider the thin, elastic, circular cylindrical shell of length L , thickness h , and radius a shown in Fig. 1. The positive directions of the axial coordinate x , the circumferential coordinate θ , the axial displacement U , the circumferential displacement V , and the normal displacement W are as indicated in Fig. 1. Donnell's partial differential equations for the vibration analysis of the cylinder can be obtained by minimizing the Lagrangian with respect to U , V , and W , and assuming the motion to be harmonic in time with a natural frequency Ω . This leads to [7]

$$\left. \begin{aligned} u_{,\xi\xi} + \left(\frac{1-\nu}{2}\right) u_{,\theta\theta} + \left(\frac{1+\nu}{2}\right) v_{,\xi\theta} + w_{,\xi} &= -\omega^2 u \\ \left(\frac{1+\nu}{2}\right) u_{,\xi\theta} + \left(\frac{1-\nu}{2}\right) v_{,\xi\xi} + v_{,\theta\theta} + w_{,\theta} &= -\omega^2 v \\ -\nu u_{,\xi} - v_{,\theta} - w - \frac{\alpha^2}{12} (w_{,\xi\xi\xi\xi} + 2w_{,\xi\xi\theta\theta} + w_{,\theta\theta\theta\theta}) &= -\omega^2 w \end{aligned} \right\} \quad (1)$$

where a subscript comma denotes differentiation, $u = U/a$,

$v = V/a$, $w = W/a$, $\xi = x/a$, $\alpha = h/a$, $\omega = \Omega a / \sqrt{E/\rho(1-\nu^2)}$, E is Young's

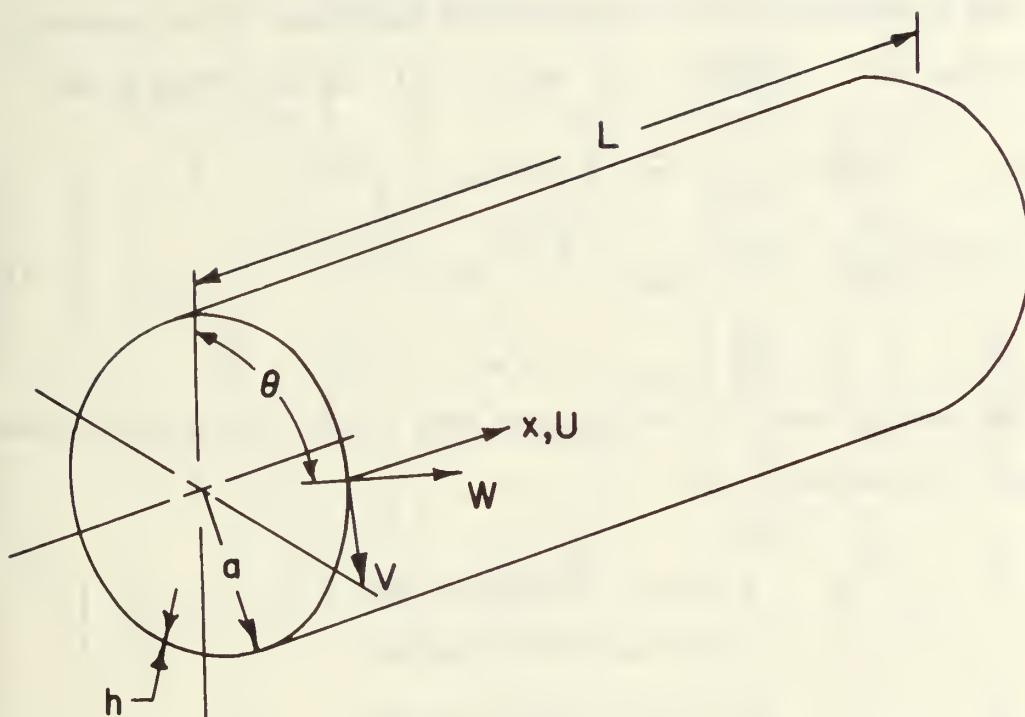


Figure 1. Geometry, Coordinates and Displacements of the Circular Cylindrical Shell

modulus, ρ is the mass density, and ν is Poisson's ratio. The boundary conditions for the simply supported cylinder are

$$u,_{\xi} = v = w = w,_{\xi\xi} = 0 \quad \xi = 0, l \quad (2)$$

where $l = L/a$.

The coordinate θ can be removed from equations (1) by assuming $u = u(\xi)\cos n\theta$, $v = v(\xi)\sin n\theta$, and $w = w(\xi)\cos n\theta$. This gives

$$\left. \begin{aligned} u,_{\xi\xi} - n^2 \left(\frac{1-\nu}{2}\right) u + n \left(\frac{1+\nu}{2}\right) v,_{\xi} + \nu w,_{\xi} &= -\omega^2 u \\ -n \left(\frac{1+\nu}{2}\right) u,_{\xi} + \left(\frac{1-\nu}{2}\right) v,_{\xi\xi} - n^2 v - n w &= -\omega^2 v \\ -\nu u,_{\xi} - n v - w - \frac{\alpha^2}{12} (w,_{\xi\xi\xi\xi} - 2n^2 w,_{\xi\xi} + n^4 w) &= -\omega^2 w \end{aligned} \right\} \quad (3)$$

The natural modes of the cylinder that satisfy the simply supported boundary conditions (2) are

$$\left. \begin{aligned} u &= \bar{u} \cos (m\pi\xi/l) \cos n\theta \\ v &= \bar{v} \sin (m\pi\xi/l) \sin n\theta \\ w &= \bar{w} \sin (m\pi\xi/l) \cos n\theta \end{aligned} \right\} \quad (4)$$

where \bar{u} , \bar{v} , and \bar{w} are arbitrary constants. Substituting equations (4) into equations (1) leads to

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ \text{sym.} & & a_{33} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{bmatrix} = -\omega^2 \begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{bmatrix} \quad (5)$$

The coefficients a_{pq} are given in Table I.

NUMERICAL MODELS

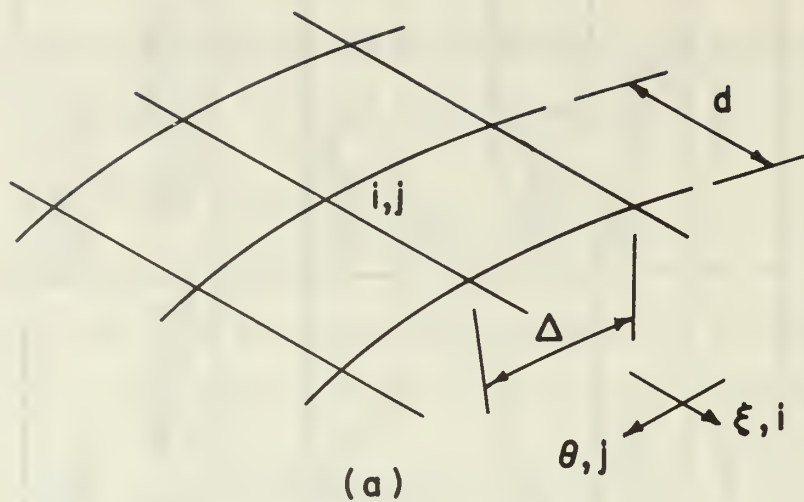
Consider the finite difference network shown in Fig. 2a, where d is the nondimensional meridional increment and Δ is the circumferential angular

TABLE I Matrix Coefficients a_{pq} , Equation (5)

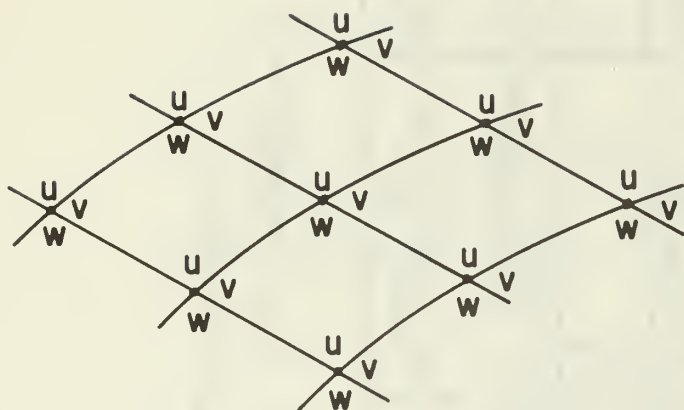
Coefficient	Differential Equations (1)	Model A	Model B	Model C	Model D
a_{11}	$-\beta_m^2 - (\frac{1-\nu}{2})n^2$	$-\beta_{ms}^2 - (\frac{1-\nu}{2})\eta_s^2$	$-\beta_{ms}^2 - (\frac{1-\nu}{2})\eta_s^2$	$-\beta_{ms}^2 - (\frac{1-\nu}{2})n^2$	$-\beta_{ms}^2 - (\frac{1-\nu}{2})n^2$
a_{12}	$(\frac{1+\nu}{2})n\beta_m$	$(\frac{1+\nu}{2})\eta_c\beta_{mc}$	$(\frac{1+\nu}{2})\eta_s\beta_{mc}$	$(\frac{1+\nu}{2})n\beta_{mc}$	$(\frac{1+\nu}{2})n\beta_{ms}$
a_{13}	$\nu\beta_m$	$\nu\beta_{mc}$	$\nu\beta_{ms}$	$\nu\beta_{mc}$	$\nu\beta_{ms}$
a_{22}	$-(\frac{1-\nu}{2})\beta_m^2 - n^2$	$-(\frac{1-\nu}{2})\beta_{ms}^2 - \eta_s^2$	$-(\frac{1-\nu}{2})\beta_{ms}^2 - \eta_s^2$	$-(\frac{1-\nu}{2})\beta_{ms}^2 - n^2$	$(\frac{1-\nu}{2})\beta_{ms}^2 - n^2$
a_{23}	$-n$	$-\eta_c$	$-\eta_s$	$-n$	$-n$
a_{33}	$-1 - \frac{\alpha^2}{12}(\beta_m^2 + n^2)$	$-1 - \frac{\alpha^2}{12}[4(\beta_{ms}^2 - \beta_{mc}^2)/d^2 + 2\beta_{ms}^2\eta_s^2 + 4(\eta_s^2 - \eta_c^2)/\Delta^2]$	$-1 - \frac{\alpha^2}{12}[4(\beta_{ms}^2 - \beta_{mc}^2)/d^2 + 2\beta_{ms}^2\eta_s^2 + 4(\eta_s^2 - \eta_c^2)/2^2]$	$-1 - \frac{\alpha^2}{12}[4(\beta_{ms}^2 - \beta_{mc}^2)/d^2 + 2\beta_{ms}^2n^2 + n^4]$	$-1 - \frac{\alpha^2}{12}[4(\beta_{ms}^2 - \beta_{mc}^2)/d^2 + 2\beta_{ms}^2n^2 + n^4]$

$$\beta_m = m\pi/\ell \quad \beta_{mc} = (\sin\beta_m d)/d \quad \beta_{ms} = (2\sin\beta_m d/2)/d$$

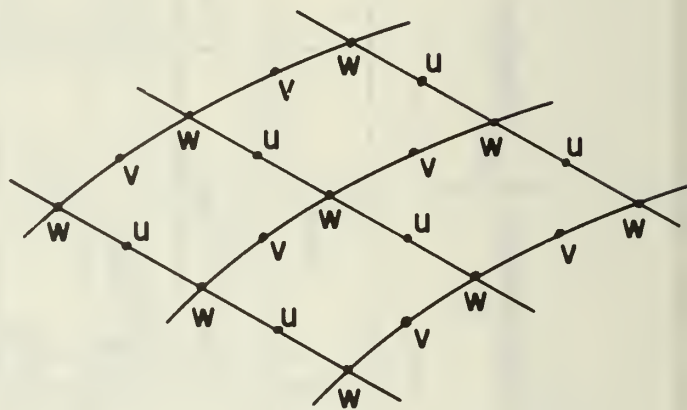
$$\eta_c = (\sin n\Delta)/\Delta \quad \eta_s = (2\sin n\Delta/2)/\Delta$$



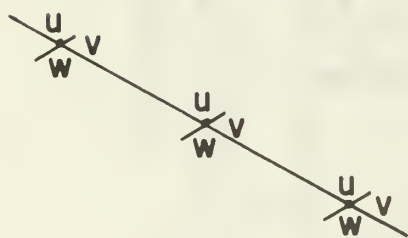
(a)



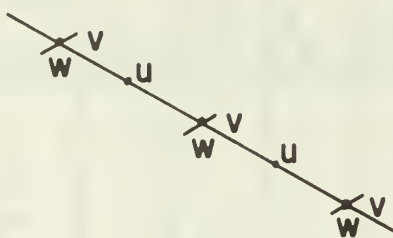
(b) Model A



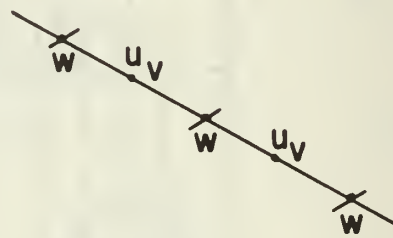
(c) Model B



(d) Model C



(e) Model D



(f) Model E

Figure 2. Numerical Models

increment. A typical station is denoted by i, j , where i and j are numbered such that $\xi_i = id$ and $\theta_j = j\Delta$. The conventional differencing scheme where u, v , and w are each defined at the same station is shown in Fig. 2b. A staggered station scheme where u, v , and w are each defined at different locations on the shell [5] is shown in Fig. 2c. The conventional scheme for the ordinary differential equations (3) is shown in Fig. 2d, and the half-station scheme corresponding to Fig. 2c is shown in Fig. 2e. The staggered scheme used in reference [2] with the energy approach is shown in Fig. 2f. These five schemes will be referred to henceforth as models A, B, C, D, and E respectively.

The difference approximations to the partial derivatives for model A are of the form

$$\left(\frac{\partial f}{\partial \xi}\right)_{i,j} = (f_{i+1,j} - f_{i-1,j})/(2d) \quad (6a)$$

$$\left(\frac{\partial^2 f}{\partial \xi^2}\right)_{i,j} = (f_{i+1,j} - 2f_{i,j} + f_{i-1,j})/d^2 \quad (6b)$$

$$\left(\frac{\partial^2 f}{\partial \xi \partial \theta}\right)_{i,j} = (f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1})/(2d)^2 \quad (6c)$$

$$\left(\frac{\partial^4 f}{\partial \xi^4}\right)_{i,j} = (f_{i+2,j} - 4f_{i+1,j} + 6f_{i,j} - 4f_{i-1,j} + f_{i-2,j})/d^4 \quad (6d)$$

$$\begin{aligned} \left(\frac{\partial^4 f}{\partial \xi^2 \partial \theta^2}\right)_{i,j} = & [f_{i+1,j+1} + f_{i-1,j+1} + f_{i+1,j-1} + f_{i-1,j-1} - 2(f_{i+1,j} + f_{i,j+1} \\ & + f_{i-1,j} + f_{i,j-1}) + 4w_{i,j}] / d^4 \end{aligned} \quad (6e)$$

The derivative approximations for model B are those given by equations (6b), (6d), (6e) and

$$\left(\frac{\partial f}{\partial \xi}\right)_{i,j} = (f_{i+\frac{1}{2},j} - f_{i-\frac{1}{2},j})/d \quad (6f)$$

$$\left(\frac{\partial^2 f}{\partial \xi \partial \theta}\right)_{i+\frac{1}{2},j} = (f_{i+1,j+\frac{1}{2}} - f_{i,j+\frac{1}{2}} - f_{i+1,j-\frac{1}{2}} + f_{i,j-\frac{1}{2}})/d^2 \quad (6g)$$

Corresponding approximations are used for the ordinary derivatives for models C and D.

Field Equations Approach

The finite difference form of equations (1) corresponding to Model A is

$$\begin{aligned} & (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/d^2 + \left(\frac{1-v}{2}\right) (u_{i,j+1} - 2u_{i,j} + u_{i,j-1})/\Delta^2 \\ & + \left(\frac{1+v}{2}\right) (v_{i+1,j+1} - v_{i-1,j+1} - v_{i+1,j-1} + v_{i-1,j-1})/(4d\Delta) + v(w_{i+1,j} - w_{i-1,j})/(2d) \\ & = -\omega^2 u_{i,j} \quad i = 0,1,2,\dots,I ; \quad j = 1,2,\dots,J \end{aligned} \quad (7a)$$

$$\begin{aligned} & \left(\frac{1+v}{2}\right) (u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1})/(4d\Delta) + \left(\frac{1-v}{2}\right) (v_{i+1,j} - 2v_{i,j} \\ & + v_{i-1,j})/d^2 + (v_{i,j+1} - 2v_{i,j} + v_{i,j-1})/\Delta^2 + (w_{i,j+1} - w_{i,j-1})/(2\Delta) \\ & = -\omega^2 v_{i,j} \quad i = 0,1,2,\dots,I ; \quad j = 1,2,\dots,J \end{aligned}$$

$$\begin{aligned}
& -v(u_{i+1,j} - u_{i-1,j})/(2d) - (v_{i,j+1} - v_{i,j-1})/(2\Delta) - w_{i,j} - \frac{\alpha^2}{12} \left\{ (w_{i+2,j} \right. \\
& - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j})/d^4 + 2 \left[w_{i+1,j+1} + w_{i-1,j+1} + w_{i+1,j-1} \right. \\
& + w_{i-1,j-1} - 2(w_{i+1,j} + w_{i,j+1} + w_{i-1,j} + w_{i,j-1}) + 4w_{i,j} \left. \right] / (d\Delta)^2 \\
& + (w_{i,j+2} - 4w_{i,j+1} + 6w_{i,j} - 4w_{i,j-1} + w_{i,j-2})/\Delta^4 \left. \right\} \\
& = -\omega^2 w_{i,j} \quad i = 1, 2, \dots, I-1 ; \quad J = 1, 2, \dots, J
\end{aligned}$$

where $I = \ell/d$ and $J = 2\pi/\Delta$. The stations $i = I+1$ are fictitious stations located one increment off the ends of the shell. The finite difference form of the boundary conditions (2) for model A is

$$\begin{aligned}
u_{i+1,j} - u_{i-1,j} = v_{i,j} = w_{i,j} = w_{i+1,j} - 2w_{i,j} + w_{i-1,j} = 0 \quad i = 0, I; \\
j = 1, 2, \dots, J
\end{aligned} \tag{7b}$$

Equations (7) constitute a set of $3(I-1)J + 12J$ homogeneous algebraic equations that define an eigenvalue problem. The eigenvalues are $-\omega^2$, and the eigenvectors are the natural modes. For this set of equations assume the modes to be

$$u_{i,j} = \bar{u} \cos(m\pi i/I) \cos(2n\pi j/J) \tag{8a}$$

$$v_{i,j} = \bar{v} \sin(m\pi i/I) \sin(2n\pi j/J) \tag{8b}$$

$$w_{i,j} = \bar{w} \sin(m\pi i/I) \cos(2n\pi j/J) \tag{8c}$$

Note that these modes satisfy the boundary conditions (7b). Furthermore,

substituting equations (8) into equations (7a) eventually leads to equations (5) with coefficients a_{pq} that are identical for every i and j defined in equations (7a). Hence, equations (8) are the correct modes for model A and would be the eigenvectors determined by a digital computer except for round-off error. The coefficients a_{pq} for this model are given in Table I.

A somewhat different set of difference equations is obtained when the half-station scheme of model B is used. The results are given in equations (A1) in the Appendix. The natural modes for this model are also given by equations (8), but with i replaced by $i+\frac{1}{2}$ in equation (8a) and j replaced by $j+\frac{1}{2}$ in equation (8b). As in the previous case, substituting equations (8) into equations (A1) leads to equations (5) for every i and j defined in equations (A1). The coefficients a_{pq} are given in Table I.

The same procedure can be followed using equations (3), where the θ dependence has been eliminated, and models C and D. The difference equations and boundary conditions for these two models are given in the Appendix as equations (A2) and (A3) respectively. The natural modes for both of these schemes are

$$u_i = \bar{u} \cos (m\pi i/I) \quad (9a)$$

$$v_i = \bar{v} \sin (m\pi i/I) \quad (9b)$$

$$w_i = \bar{w} \sin (m\pi i/I) \quad (9c)$$

except in model D, where i is replaced by $i+\frac{1}{2}$ in equation (9a). The coefficients a_{pq} for both models are given in Table I.

Equations (A2) are analogous to the set of difference equations developed by the computer program SALORS [3]. The two sets of equations are not identical because Sanders' shell theory is used in SALORS, and

because the governing equations are four second order differential equations in terms of u , v , w , and m_ξ , the nondimensional meridional bending moment. However, it can be shown that by eliminating the discrete variable $m_{\xi i}$ from the difference equations, the four equation formulation used in SALORS reduces to the identical three equation formulation given by equations (A2), when Donnell's theory is used.

Energy Approach

Sets of difference equations can also be developed by equating to zero the variation of the finite difference form of the Lagrangian with respect to the discrete displacement variables and the Lagrangian multipliers. The Lagrangian L can be expressed in the form

$$L = \Pi - T + \sum_{k=1}^8 \lambda_k \delta_k \quad (10)$$

where Π is the shell strain energy, T is the shell kinetic energy, λ_k is a Lagrangian multiplier, and δ_k is a constraint condition. Chuang and Veletsos [5] have shown that when model A is used the difference equations developed from the energy approach are not the same as equations (7a), which were developed from the field equation approach. However, when model B is used, the two approaches yield identical equations. This section will consider the development of the difference equations by the energy approach using the models C, D, and E where the derivatives with respect to θ have been replaced with multiples of n .

The shell strain energy in the Donnell theory in the n^{th} mode can be shown to be

$$\Pi = \frac{\pi^2 a E h}{2(1-\nu^2)} \int_0^\ell (\Pi_A + \Pi_B) d\xi \quad (11a)$$

where

$$\begin{aligned}\Pi_A &= (u, \xi)^2 + 2\nu (nv-w) u, \xi + (nv-w)^2 \\ &\quad + \frac{\alpha^2}{12} [(w, \xi\xi)^2 - 2\nu n^2 w w, \xi\xi + n^4 w^2] \\ \Pi_B &= \left(\frac{1-\nu}{2}\right) (v, \xi - nu)^2 + \frac{\alpha^2}{6} (1-\nu) n^2 (w, \xi)^2\end{aligned}$$

The shell kinetic energy in the n^{th} mode is taken as

$$T = \frac{\pi a^2 E h}{2(1-\nu^2)} \int_0^{\ell} (T_A + T_B) d\xi \quad (11b)$$

where

$$\begin{aligned}T_A &= (v, \tau)^2 + (w, \tau)^2 \\ T_B &= (u, \tau)^2\end{aligned}$$

and $\tau = t \sqrt{E/\rho(1-\nu^2)}/a$ where t is time.

Consider first model C. The integrals in equation (11) can be approximated by

$$\Pi = \frac{\pi a^2 E h}{2(1-\nu^2)} \sum_{i=0}^I (\Pi_A + \Pi_B)_i d_i \quad (12a)$$

$$T = \frac{\pi a^2 E h}{2(1-\nu^2)} \sum_{i=0}^I (T_A + T_B)_i d_i \quad (12b)$$

where $(\Pi_A + \Pi_B)_i$ and $(T_A + T_B)_i$ represent the strain and kinetic energy expressed at the i^{th} station respectively, and $d_i = d$ except at $i = 0$ and $i = I$ where $d_i = d/2$. The constraint conditions for the simply supported shell are $v_0 = w_0 = v_I = w_I = 0$. Carrying out the variation of the Lagrangian

with respect to the variables $u_{-1}, v_{-1}, w_{-1}, u_0, v_0, \dots, v_{I+1}, w_{I+1}, \lambda_1, \lambda_2, \lambda_3$, and λ_4 and assuming harmonic motion leads to equations (A4) in the Appendix. A comparison of equations (A4) with equations (A2) reveals several differences. In the recursion equations, the differences appear in the approximation to the second derivative, and the energy approach introduces the additional term $(\alpha^2 v n^2 d^2 / 24)(w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2})/d^4$ in equation (A4c). Perhaps a more significant difference is the fact that the five special equations (A4e) - (A4i) are dissimilar to the equations obtained by applying the boundary conditions (A4d) to equations (A4a) - (A4b) at $i=0$ and $i=1$. The consequence of this is that the modes given by equations (9) are not the eigenvectors of equations (A4). The fourth boundary condition in equation (A4d) also eliminates equation (9c) as the mode for w .

For model D, Π_A and T_A are evaluated at i , as in the previous scheme, but Π_B and T_B are evaluated at $i+\frac{1}{2}$. Hence

$$\Pi = \frac{\pi a^2 E h}{2(1-\nu^2)} \left[\sum_{i=0}^I \Pi_{A_i} d_i + \sum_{i=0}^{I-1} \Pi_{B_{i+\frac{1}{2}}} d \right] \quad (13a)$$

$$T = \frac{\pi a^2 E h}{2(1-\nu^2)} \left[\sum_{i=0}^I T_{A_i} d_i + \sum_{i=0}^{I-1} T_{B_{i+\frac{1}{2}}} d \right] \quad (13b)$$

The constraints are $v_0 = w_0 = v_I = w_I = 0$. Performing the variation of the Lagrangian and assuming harmonic motion yield a set of difference equations identical to equations (A1).

For model E, equations (12) are used in conjunction with equations (6f) for $(u, \xi)_i$ and $(v, \xi)_i$; equation (6a) for $(w, \xi)_i$; equation (6b) for $(w, \xi \xi)_i$; $u_i = (u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}})/2$, and $v_i = (v_{i+\frac{1}{2}} + v_{i-\frac{1}{2}})/2$. The difference

equations for this scheme are given by equations (A5) in the Appendix.

Note that, as is the situation with model C, the three special equations (A5f) - (A5h) are dissimilar to the equations obtained by applying the boundary conditions (A5d) to equations (A5a) - (A5c). Hence, the modes given by equations (9) are not the eigenvectors of model E.

COMPARISON

The natural frequencies from the differential equations and models A-D have been computed for the four shells $\alpha = .01, \ell = 2$; $\alpha = .01, \ell = 10$; $\alpha = .1, \ell = 2$; and $\alpha = .1, \ell = 10$; for several values of m and n . Three axial increment sizes $d = \ell/20, \ell/40$, and $\ell/100$; and two circumferential increment sizes $\Delta = 2\pi/10$ and $2\pi/20$ were used. Poisson's ratio was .3. Since there are three degrees of freedom per mode, there are three eigenvalues for each value of m and n . The results for the lowest of the three natural frequencies for the two shells $\alpha = .01, \ell = 10$, and $\alpha = .1, \ell = 2$ are given in Tables IIA and IIB respectively. Similar results were obtained for the other two shells. The natural frequencies from three dimensional elasticity theory are also given in Tables IIA and IIB [8]. Rather than present the natural frequencies of the numerical models, the quantity $100(1-\omega/\omega_e)$ is presented in Tables IIA and IIB where ω_e is the natural frequency associated with the differential equations (1). Thus, the smaller the number, the more accurate the natural frequency.

Two other measures of the accuracy of the numerical models have been computed. These measures are

$$ERMN = 100 \frac{\sum_{m=1}^{m_{\max}} \sum_{n=0}^{n_{\max}} (1-\omega/\omega_e)}{(m_{\max} n_{\max})}$$

TABLE IIA Natural Frequencies

$\alpha = .01$

$\ell = 10$

Numerical Models, 100(1-w/w _e)											
m	n	3-D Elast. Theory	w _e Diff. Eq. (1)	A			B		C	D	ℓ/d
				Δ=2π/10	Δ=2π/20	Δ=2π/10	Δ=2π/20				
1	0		.186	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	20 40 100
1	4	.0426	.0454	-1980 -1980 -1980	-1170 -1170 -1170	40.0 40.0 40.0	12.0 12.0 12.0	- .5 0 0	0 0 0	0 0 0	20 40 100
1	9		.233	- 326 - 326 - 326	- 326 - 326 - 326		51.5 51.5 51.5	0 0 0	0 0 0	0 0 0	20 40 100
9	0		.948	- 2.0 - .5 0	- 2.0 - .5 0	0 0 0	0 0 0	- 2.0 - .5 0	0 0 0	0 0 0	20 40 100
9	4		.316	- 199 - 197 - 196	- 107 - 109 - 110	-23.0 -32.0 -35.0	3.5 - 5.0 - 7.5	- 1.0 - .5 0	10.5 3.0 .5	20 40 100	
9	9		.269	- 268 - 266 - 265	- 268 - 266 - 265		29.5 24.5 23.0	- 4.5 - 1.5 0	2.5 .5 0	20 40 100	
19	0		.958	- 4.5 - 2.0 - .5	- 4.5 - 2.0 - .5	.5 0 0	.5 0 0	- 4.5 - 2.0 - .5	.5 0 0	20 40 100	
19	4		.668	- 45.0 - 44.0 - 43.0	- 8.5 - 18.5 - 21.0	11.0 - 9.0 -13.0	24.5 2.5 - 2.5	-24.0 - 6.0 - 1.0	29.0 6.5 1.0	20 40 100	
19	9		.443	- 126 - 122 - 121	- 126 - 122 - 121		29.5 - 1.5 - 8.0	15.5 2.0 0	28.0 8.5 1.5	20 40 100	

Note: The value 0 is used when $100(1-w/w_e) < .25$

TABLE IIB Natural Frequencies

$$\ell = 2$$

$$\alpha = .1$$

m	n	3-D Elast. Theory	ω_e Diff. Eq.(1)	Numerical Models, $100(1-\omega/\omega_e)$					
				A		B		C	D
				$\Delta=2\pi/10$	$\Delta=2\pi/20$	$\Delta=2\pi/10$	$\Delta=2\pi/20$		
1	0	.930	.929	0	0	0	0	0	0
				0	0	0	0	0	0
				0	0	0	0	0	0
1	4	.486	.530	-87.5	-43.5	30.0	10.0	0	0
				-87.5	-43.5	30.0	10.0	0	0
				-87.5	-43.5	30.0	10.0	0	0
1	9		2.39		35.0		50.0	0	0
					35.0		50.0	0	0
					35.0		50.0	0	0
9	0		5.85	15.0	15.0	15.0	15.0	15.0	15.0
				4.0	4.0	4.0	4.0	4.0	4.0
				.5	.5	.5	.5	.5	.5
9	4		6.28	17.0	15.0	17.0	15.0	14.0	14.0
				6.5	4.5	6.5	4.5	3.5	3.5
				3.5	1.0	4.5	1.0	.5	.5
9	9		8.10		25.0		25.5	11.0	11.0
					17.0		17.0	3.0	3.0
					14.5		15.0	.5	.5
19	0		17.7	35.0	35.0	35.0	35.0	35.0	35.0
				9.0	9.0	9.0	9.0	9.0	9.0
				1.5	1.5	1.5	1.5	1.5	1.5
19	4		17.8	33.5	33.0	34.5	33.5	33.0	33.0
				8.5	8.0	9.0	9.0	8.0	9.0
				1.0	1.0	1.5	1.5	1.0	1.0
19	9		18.4		31.0		33.5	25.0	30.0
					6.5		10.5	4.5	8.0
					.5		3.5	.5	1.0

Note: The value 0 is used when $100(1-\omega/\omega_e) < .25$

$$\text{WERMN} = 100 \sum_{m=1}^{m_{\max}} \sum_{n=0}^{n_{\max}} (1 - \omega_{mn}/\omega_e) / (mn+m) \quad / \quad (m_{\max} n_{\max})$$

where $m_{\max} = \ell/d$ and $n_{\max} = \pi/\Delta$. The two error measures are given in Tables IIIA and IIIB for the three natural frequencies from each of the numerical models, with $n_{\max} = 5$ and 10 and $m_{\max} = 20, 40$, and 100. Similar results were obtained for the other two shells.

TABLE IIIA Measure of Error

$$\alpha = .01 \quad \ell = 2$$

		Numerical Models											
		A			B			C			D		
ℓ/a		20	40	100	20	40	100	20	40	100	20	40	100
$\Delta=2\pi/10$	ERMN	30	29	25	17	23	23	15	21	23	16	22	23
		13	13	13	15	14	13	13	13	13	13	13	13
		16	14	13	15	14	13	14	13	13	13	13	13
$n_{\max}=5$	WERMN	321	174	72	71	49	21	43	35	16	46	35	16
		49	25	10	58	29	11	43	22	9	45	22	9
		69	34	13	59	29	12	47	23	9	45	22	9
$\Delta=2\pi/20$	ERMN	44	37	29	18	23	23	16	21	23	16	22	23
		12	13	13	16	15	14	10	12	13	13	13	13
		18	16	14	16	15	14	14	13	13	13	13	13
$n_{\max}=10$	WERMN	313	166	68	43	30	13	30	22	10	30	23	10
		34	18	8	42	21	8	25	14	6	28	14	6
		49	24	10	42	21	8	30	15	6	28	14	6

$$\text{ERMN} = 100 \sum_{m=1}^{m_{\max}} \sum_{n=0}^{n_{\max}} (1 - \omega/\omega_e)/(m_{\max} n_{\max})$$

$$\text{WERMN} = 100 \sum_{m=1}^{m_{\max}} \sum_{n=0}^{n_{\max}} (1 - \omega/\omega_e)/((mn + m)/(m_{\max} n_{\max}))$$

TABLE IIIB Measure of Error

$\alpha = .1$ $\ell = 10$

Numerical Models												
ℓ/d		A			B			C			D	
		20	40	100	20	40	100	20	40	100	20	40
$\Delta=2\pi/10$	ERMIN	44	37	26	16	22	19	11	18	17	12	19
		9	11	18	15	15	19	7	9	17	10	12
		20	18	15	16	15	14	14	14	13	11	12
$n_{\max}=5$	WERMIN	592	315	128	71	50	22	29	28	13	33	29
		49	29	16	67	36	17	31	18	11	38	21
		85	44	18	66	35	15	45	23	9	38	21
$\Delta=2\pi/20$	ERMIN	23	26	21	25	27	21	9	16	16	10	16
		9	9	18	16	16	21	5	7	15	7	10
		20	21	17	16	16	15	10	13	13	7	10
$n_{\max}=10$	WERMIN	153	88	37	60	39	17	18	17	8	20	18
		33	19	11	44	25	12	18	10	7	21	12
		52	29	12	44	24	10	26	14	6	21	12

$$\text{ERMIN} = 100 \sum_{m=1}^{m_{\max}} \sum_{n=0}^{n_{\max}} (1 - \omega/\omega_e)/(m_{\max} n_{\max})$$

$$\text{WERMIN} = 100 \sum_{m=1}^{m_{\max}} \sum_{n=0}^{n_{\max}} (1 - \omega/\omega_e)/(mn + m)/(m_{\max} n_{\max})$$

CONCLUSIONS

An examination of the results given in Tables IIA, IIB, IIIA, and IIIB reveals the following:

- (1) Model A yields extremely inaccurate natural frequencies for all m and $n > 0$ for the longer, thinner cylinder. The results are particular poor for $m = 1$. The frequencies are more accurate for the shorter, thicker cylinder.
- (2) Model B yields considerably more accurate natural frequencies than model A for the longer, thinner shell, and about equally accurate frequencies for the shorter thicker cylinder, except for $m = 1$ where model B is the more accurate. In general, model B is the more accurate model for all four shells.
- (3) Models C and D yield essentially the same natural frequencies for all modes for all four shells. In some instances, model C is slightly more accurate, and in others, model D is the more accurate. In general, models C and D yield more accurate frequencies than models A and B.

Hence, model B is preferred over model A, and models C and D are preferred over models A and B, provided the field equation approach is used to develop the difference equations for model C. The conclusion that model B is more accurate than model A is in agreement with the results presented in reference (5). The conclusion that models C and D are of equal accuracy is somewhat surprising, in view of the fact that model D yields identical difference equations from both the field equation and the energy approach.

The staggered station schemes of models B and D appear to be ideally suited to Donnell's theory with the boundary conditions given by equation (2) because only even or odd ordered derivatives of the dependent variables appear in any one field or boundary equation. General shells of revolution with meridional curvature do not have the consistent even and odd ordering of derivatives. Hence, the staggered station schemes do not appear to be more suitable for the general shell of revolution than for the cylinder. On the other hand, the meridional curvature may have some influence.

REFERENCES

1. Hartung, R. F., "An Assessment of Current Capability for Computer Analysis of Shell Structures" Air Force Flight Dynamics Laboratory, Technical Report AFFDL-TR-publication pending.
2. Bushnell, D., "Analysis of Buckling and Vibration of Ring-stiffened, Segmented Shells of Revolution," Int. J. Solids and Structures 6 157-181 (January, 1970).
3. Anderson, M. S., et al., "Stress, Buckling, and Vibration Analysis of Shells of Revolution " presented at the Computer-oriented Analysis of Shell Structures Conference, Palo Alto, Calif. (August, 1970).
4. Cyrus, N. J. and Fulton, R. E., "Finite Difference Accuracy in Structural Analysis," J. St. Div. ASCE 92 459-471 (December, 1966).
5. Chuang, K. P. and Veletsos, A. S., "A Study of Two Approximate Methods of Analyzing Cylindrical Shell Roofs," Civil Eng. Studies, St. Res. Series, No. 258, U. of Ill. (October, 1962).
6. Ball, R. E , "Dynamic Analysis of Rings by Finite Differences," J. Eng. Mech. Div. ASCE 93 1-10 (February 1967).
7. Kraus, H., Thin Elastic Shells, John Wiley & Sons, New York, (1967), p. 297.
8. Armenakas, A. E., Free Vibrations of Circular Cylindrical Shells, Pergamon Press, Oxford, 1969.

APPENDIX

Difference Equations for Model B

The equations at a typical station are

$$\begin{aligned}
 & (u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j})/d^2 + \left(\frac{1-v}{2}\right) (u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} \\
 & + u_{i+\frac{1}{2},j-1})/\Delta^2 + \left(\frac{1+v}{2}\right) (v_{i+1,j+\frac{1}{2}} - v_{i,j+\frac{1}{2}} - v_{i+1,j-\frac{1}{2}} + v_{i,j-\frac{1}{2}})/(d\Delta) \\
 & + v(w_{i+1,j} - w_{i,j})/d = -\omega^2 u_{i+\frac{1}{2},j} \quad i = 0,1,2,\dots,I-1 ; \quad j = 1,2,\dots,J \quad (A1a)
 \end{aligned}$$

$$\begin{aligned}
 & \left(\frac{1+v}{2}\right)(u_{i+\frac{1}{2},j+1} - u_{i-\frac{1}{2},j+1} - u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j})/(d\Delta) + \left(\frac{1-v}{2}\right) (v_{i+1,j+\frac{1}{2}} \\
 & - 2v_{i,j+\frac{1}{2}} + v_{i-1,j+\frac{1}{2}})/d^2 + (v_{i,j+\frac{3}{2}} - 2v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}})/\Delta^2 \\
 & + (w_{i,j+1} - w_{i,j})/\Delta = -\omega^2 v_{i,j+\frac{1}{2}} \quad i = 1,2,\dots,I-1 ; \quad j = 1,2,\dots,J \quad (A1b)
 \end{aligned}$$

$$\begin{aligned}
 & -v(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j})/d - (v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}})/\Delta - w_{i,j} - \frac{d^2}{12} \left\{ (w_{i+2,j} \right. \\
 & - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j})/d^4 + 2 \left[w_{i+1,j+1} + w_{i-1,j+1} + w_{i+1,j-1} \right. \\
 & + w_{i-1,j-1} - 2(w_{i+1,j} + w_{i,j+1} + w_{i-1,j} + w_{i,j-1}) + 4w_{i,j} \left. \right] / (d\Delta)^2 \\
 & + w_{i,j+2} - 4w_{i,j+1} + 6w_{i,j} - 4w_{i,j-1} + w_{i,j-2})/\Delta^4 \left. \right\} \\
 & = -\omega^2 w_{i,j} \quad i = 1,2,\dots,I-1 ; \quad j = 1,2,\dots,J \quad (A1c)
 \end{aligned}$$

The boundary conditions are

$$u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j} = v_{i,j+\frac{1}{2}} = w_{i,j} = w_{i+1,j} - 2w_{i,j} + w_{i-1,j} = 0$$

$$i = 0, I ; j = 1, 2, \dots, J \quad (\text{A1d})$$

Difference Equations for Model C

The equations at a typical station are

$$(u_{i+1} - 2u_i + u_{i-1})/d^2 - n^2 \left(\frac{1-v}{2}\right) u_i + n \left(\frac{1+v}{2}\right) (v_{i+1} - v_{i-1})/(2d)$$

$$+ v(w_{i+1} - w_{i-1})/(2d) = -\omega^2 u_i \quad i = 0, 1, 2, \dots, I ; n = 0, 1, 2, \dots \quad (\text{A2a})$$

$$-n \left(\frac{1+v}{2}\right) (u_{i+1} - u_{i-1})/(2d) + \left(\frac{1-v}{2}\right) (v_{i+1} - 2v_i + v_{i-1})/d^2 - n^2 v_i$$

$$-nw_i = -\omega^2 v_i \quad i = 0, 1, 2, \dots, I ; n = 0, 1, 2, \dots \quad (\text{A2b})$$

$$-v(u_{i+1} - u_{i-1})/(2d) - nv_i - w_i - \frac{\alpha^2}{12} (w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2})$$

$$/d^4 - 2n^2 (w_{i+1} - 2w_i + w_{i-1})/d^2 + n^4 w_i$$

$$= -\omega^2 w_i \quad i = 1, 2, \dots, I-1 ; n = 0, 1, 2, \dots \quad (\text{A2c})$$

The boundary conditions are

$$u_{i+1} - u_{i-1} = v_i = w_i = w_{i+1} - 2w_i + w_{i-1} = 0$$

$$i = 0, I \quad (\text{A2d})$$

Difference Equations for Model D

The equations at a typical station are

$$(u_{i+\frac{3}{2}} - 2u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}})/d^2 - n^2 \left(\frac{1-v}{2}\right) u_{i+\frac{1}{2}} + n \left(\frac{1+v}{2}\right) (v_{i+1} - v_i)/d \quad (A3a)$$

$$+ v (w_{i+1} - w_i)/d = - \omega^2 u_{i+\frac{1}{2}} \quad i = 0, 1, 2, \dots, I-1 ; n = 0, 1, 2, \dots$$

$$- n \left(\frac{1+v}{2}\right) (u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}})/d + \left(\frac{1-v}{2}\right) (v_{i+1} - 2v_i + v_{i-1})/d^2 - n^2 v_i \quad (A3b)$$

$$- n w_i = - \omega^2 v_i \quad i = 1, 2, \dots, I-1 ; n = 0, 1, 2, \dots$$

$$- v (u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}})/d - n v_i - w_i - \frac{\alpha^2}{12} \left[(w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2})/d^4 - 2n^2 (w_{i+1} - 2w_i + w_{i-1})/d^2 + n^4 w_i \right] \quad (A3c)$$

$$= - \omega^2 w_i \quad i = 1, 2, \dots, I-1 ; n = 0, 1, 2, \dots$$

The boundary conditions are

$$u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} = v_i = w_i = w_{i+1} - 2w_i + w_{i-1} = 0$$

$$i = 0, I \quad (A3d)$$

Difference equations for Model C, Energy Approach

The equations at a typical station are

$$(u_{i+2} - 2u_i + u_{i-2})/(2d)^2 - n^2 \left(\frac{1-v}{2}\right) u_i + n \left(\frac{1+v}{2}\right) (v_{i+1} - v_{i-1})/(2d) + v(w_{i+1} - w_{i-1})/(2d) = -\omega^2 u_i \quad i = 2, 3, \dots, I-2 ; \quad n = 0, 1, 2, \dots \quad (A4a)$$

$$- n \left(\frac{1+v}{2}\right) (u_{i+1} - u_{i-1})/(2d) + \left(\frac{1-v}{2}\right) (v_{i+2} - 2v_i + v_{i-2})/(2d)^2 - n^2 v_i - n w_i = -\omega^2 v_i \quad i = 2, 3, \dots, I-2 ; \quad n = 0, 1, 2, \dots \quad (A4b)$$

$$- v (u_{i+1} - u_{i-1})/(2d) - n v_i - w_i - \frac{\alpha^2}{12} \left[(1 + v n^2 d^2 / 2) (w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2})/d^4 - 2n^2 (w_{i+2} - 2w_i + w_{i-2})/(2d)^2 + n^4 w_i \right] = -\omega^2 w_i \quad i = 2, 3, \dots, I-2 ; \quad n = 0, 1, 2, \dots \quad (A4c)$$

The equations corresponding to the boundary conditions at the initial end are

$$u_1 - u_{-1} = v_0 = w_0 = w_1 + w_{-1} - (1-\nu)n^2d^2 (w_1 - w_{-1})/2 = 0 \quad (A4d)$$

There are five special equations

$$(u_2 - u_0)/(2d)^2 + \nu n v_1/(2d) + \nu w_1/(2d) = -\omega^2 u_0/2 \quad (A4e)$$

$$(v_1 - v_{-1})/(2d) - n u_0 = 0 \quad (A4f)$$

$$\begin{aligned} (u_3 - u_1)/(2d)^2 - n^2 \left(\frac{1-\nu}{2}\right) u_1 + n \left(\frac{1+\nu}{2}\right) v_2/(2d) \\ + \nu w_2/(2d) = -\omega^2 u_1 \end{aligned} \quad (A4g)$$

$$\begin{aligned} -n \left(\frac{1+\nu}{2}\right) (u_2 - u_0) - n \left(\frac{1-\nu}{2}\right) u_0/(2d) + \left(\frac{1-\nu}{2}\right) (v_3 - v_1)/(2d) \\ + n^2 v_1 - n w_1 = -\omega^2 v_1 \end{aligned} \quad (A4h)$$

$$\begin{aligned} -\nu (u_2 - u_0)/(2d) - n v_1 - w_1 - \frac{\alpha^2}{12} \left\{ \left[w_3 - 4w_2 + (11/2 - \gamma) w_1 \right] / d^4 \right. \\ + (\nu n^2 d^2 / 2) \left[w_3 - 4w_2 + (13/2 - \gamma) w_1 \right] / d^4 - 2n^2 \left[w_3 \right. \\ \left. - (3/2 + \gamma) w_1 \right] / (2d)^2 + n^4 w_1 \left. \right\} = -\omega^2 w_1 \end{aligned} \quad (A4i)$$

where

$$\gamma = \frac{1}{2} \left[2 - (1-\nu) n^2 d^2 \right] / \left[2 + (1-\nu) n^2 d^2 \right]$$

Similar equations apply at the final end of the cylinder.

Difference Equations for Model E, Energy Approach

The equations at a typical station are

$$\begin{aligned} & (u_{i+\frac{3}{2}} - 2u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}})/d^2 - n^2 \left(\frac{1-v}{2}\right) (u_{i+\frac{3}{2}} + 2u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}})/4 \\ & + n \left(\frac{1+v}{2}\right) (v_{i+\frac{3}{2}} - v_{i-\frac{1}{2}})/(2d) + v (w_{i+1} - w_i)/d = \\ & - \omega^2 (u_{i+\frac{3}{2}} + 2u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}})/4 \end{aligned}$$

$$i = 1, 2, \dots, I-2 \quad n = 0, 1, 2, \dots \quad (A5a)$$

$$\begin{aligned} & -n \left(\frac{1+v}{2}\right) (u_{i+\frac{3}{2}} - u_{i-\frac{1}{2}})/(2d) + \left(\frac{1-v}{2}\right) (v_{i+\frac{3}{2}} - 2v_{i+\frac{1}{2}} + v_{i-\frac{1}{2}})/d^2 \\ & - n^2 (v_{i+\frac{3}{2}} + 2v_{i+\frac{1}{2}} + v_{i-\frac{1}{2}})/4 - n(w_{i+1} + w_i)/2 = -\omega^2 (v_{i+\frac{3}{2}} + 2v_{i+\frac{1}{2}} + v_{i-\frac{1}{2}})/4 \end{aligned}$$

$$i = 1, 2, \dots, I-2 \quad n = 0, 1, 2, \dots \quad (A5b)$$

$$-v(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}})/d - n(v_{i+\frac{1}{2}} + v_{i-\frac{1}{2}})/2 - w_i - \frac{\alpha^2}{12} (1+vn^2d^2/2) (w_{i+2}$$

$$-4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2})/d^4 - 2n^2 (w_{i+2} - 2w_i + w_{i-2})/(2d)^2$$

$$+ n^4 w_i = -\omega^2 w_i \quad i = 2, 3, \dots, I-2 \quad n = 0, 1, 2, \dots \quad (A5c)$$

The equations corresponding to the boundary conditions at the initial end are

$$v_{\frac{1}{2}} + v_{-\frac{1}{2}} = w_0 = w_1 + w_{-1} - (1-\nu) n^2 d^2 (w_1 - w_{-1})/2 = 0 \quad (A5d)$$

$$\begin{aligned} (u_{\frac{1}{2}} - u_{-\frac{1}{2}})/d^2 - n^2 \left(\frac{1-\nu}{2}\right) (u_{\frac{1}{2}} + u_{-\frac{1}{2}})/4 + n\left(\frac{1-\nu}{2}\right) v_{\frac{1}{2}}/d \\ = -\omega^2 (u_{\frac{1}{2}} + u_{-\frac{1}{2}})/4 \end{aligned} \quad (A5e)$$

There are three special equations

$$\begin{aligned} (u_{\frac{3}{2}} - \frac{3}{2}u_{\frac{1}{2}} + \frac{1}{2}u_{-\frac{1}{2}})/d^2 - n^2 \left(\frac{1-\nu}{2}\right) (u_{\frac{3}{2}} + \frac{3}{2}u_{\frac{1}{2}} + \frac{1}{2}u_{-\frac{1}{2}})/4 \\ + n \left(\frac{1+\nu}{2}\right) (v_{\frac{3}{2}} + v_{\frac{1}{2}})/(2d) - n \left(\frac{1-\nu}{2}\right) v_{\frac{1}{2}}/(2d) + \nu w_1/d = \\ - \omega^2 (u_{\frac{3}{2}} + \frac{3}{2}u_{\frac{1}{2}} + \frac{1}{2}u_{-\frac{1}{2}}) \end{aligned} \quad (A5f)$$

$$-n\left(\frac{1+\nu}{2}\right) (u_{\frac{3}{2}} - u_{-\frac{1}{2}})/(2d) + \nu n (u_{\frac{1}{2}} - u_{-\frac{1}{2}})/(2d) + \left(\frac{1-\nu}{2}\right)$$

$$(v_{\frac{3}{2}} - v_{\frac{1}{2}})/d^2 - n^2 (v_{\frac{3}{2}} + v_{\frac{1}{2}})/4 + n w_1/2 = -\omega^2 (v_{\frac{3}{2}} + v_{\frac{1}{2}})/4 \quad (A5g)$$

$$\begin{aligned} -\nu(u_{\frac{3}{2}} - u_{\frac{1}{2}})/d - n(v_{\frac{3}{2}} + v_{\frac{1}{2}})/2 - w_1 - \frac{\alpha^2}{12} \left\{ \left[w_3 - 4w_2 + (11/2 - \gamma) w_1 \right] /d^4 \right. \\ \left. + (\nu n^2 d^2/2) \left[w_3 - 4w_2 + (13/2 - \gamma) w_1 \right] /d^4 - 2n^2 \left[w_3 - (3/2 + \gamma) w_1 \right] / (2d)^2 \right. \\ \left. + n^4 w_1 \right\} = -\omega^2 w_1 \end{aligned}$$

Similar equations apply at the final end of the cylinder.

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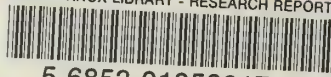
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1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP	
3. REPORT TITLE Vibration Analysis of Cylindrical Shells by Several Finite Difference Schemes			
4. DESCRIPTIVE NOTES (Type of report and, inclusive dates) Technical Report, 1971			
5. AUTHOR(S) (First name, middle initial, last name) Robert E. Ball			
6. REPORT DATE		7a. TOTAL NO. OF PAGES 35	7b. NO. OF REFS 8
8a. CONTRACT OR GRANT NO. NPS Foundation Research Program		9a. ORIGINATOR'S REPORT NUMBER(S) NPS-57Bp7161A	
b. PROJECT NO.			
c.		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d.			
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California 93940	
13. ABSTRACT Several finite differencing schemes are used to compute the natural frequencies and mode shapes of simply supported circular cylindrical shells in an attempt to determine the most accurate numerical mode. Sets of difference equations are developed from the governing differential field equations and by minimizing the finite difference form of the Lagrangian energy function. Both finite differences and trigonometric expansions are used to model the circumferential behavior. Staggered or half-stations are used in addition to the conventional differencing schemes. The results indicate that the schemes using the trigonometric expansions are generally more accurate than those using finite differences for the circumferential derivatives. Furthermore, the conventional differencing scheme is shown to be as accurate as the half-station scheme when the field equation approach is used in conjunction with the trigonometric expansions.			

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